Beta and Gamma functions of Cayley-Dickson numbers

S.V. Ludkovsky

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1 Introduction.

This paper continuous investigations of function theory over Cayley-Dickson algebras [12, 13]. Cayley-Dickson algebras $\mathcal{A}_v$ over the field of real numbers coincide with the field $\mathbf{C}$ of complex numbers for $v = 1$, with the skew field of quaternions, when $v = 2$, with the division nonassociative noncommutative algebra $\mathbf{K}$ of octonions for $v = 3$, for each $v \ge 4$ they are nonassociative and not division algebras. The algebra $\mathcal{A}_{v+1}$ is obtained from $\mathcal{A}_v$ with the help of the doubling procedure. This work provides examples of $\mathcal{A}_v$-meromorphic functions and usages of line integrals over $\mathcal{A}_v$. Here notations of previous papers [12, 13] are used. Discussions of references and results of others authors can be found in [12, 13] as well as physical applications (see also [1, 4, 6, 7, 9, 10, 11, 14, 15] and references therein). Beta and Gamma functions illustrate general theory of meromorphic functions of Cayley-Dickson numbers and also applications of line integration over $\mathcal{A}_v$.

The results below show some similarity with the complex case and as well differences caused by noncommutativity and nonassociativity of Cayley-Dickson algebras. It is necessary to mention that before works [12, 13] there was not any publication of others authors devoted to the line integration of continuous functions of Cayley-Dickson numbers or even quaternions along rectifiable paths. In works of others authors integrations over submanifolds of codimension 1 in $\mathbf{H}$ or $\mathbf{K}$ were used instead of line integral. Therefore, in this respect publications [12, 13] are the first devoted to (integral) holomorphic functions of Cayley-Dickson numbers.

If g is a complex holomorphic function on a domain V in the complex plane Π embedded into $\mathcal{A}_v$ and g has a local expansion $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in the ball $B(\Pi, z_0, r^-)$ for each $z_0 \in Int(V)$, where $Int(V)$ is the interior of V in Π, $r > 0$ and $a_n = a_n(z_0) \in \mathbf{C}$ may depend on parameter z_0, $B(X, a, r^-) := \{x \in X : \rho(x, a) < r\}$ for a metrizable space X with metric ρ, f is a function on a domain U in $\mathcal{A}_v$, $v \ge 2$, such that $V \subset U \cap \Pi$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the local expansion of f in $B(\mathcal{A}_v, z_0, r^-)$ for each $z_0 \in V$, then the line integral $\int_\omega f(z)dz$ over $\mathcal{A}_v$ for a rectifiable path ω in V coincides with the classical complex Cauchy line integral, since $\hat{f}|_V = f|_V$. Nevertheless, if f is an $\mathcal{A}_v$-holomorphic function on a domain U in $\mathcal{A}_v$, $V = U \cap \Pi$ is a domain in the complex plane Π embedded into $\mathcal{A}_v$, then in general $\int_\omega f(z)dz$ can not be reduced to Cauchy line integral for any rectifiable path ω in V, since the generalized operator $\hat{f}$ is defined by values of f in the neighbourhood of ω (see [12, 13]).

2 Beta and Gamma functions of Cayley-Dickson numbers.

1. Definition. The Gamma function is defined by the formula:

(1) $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt$,

whenever this (Eulerian of the second kind) integral converges and defined by $\mathcal{A}_v$-holomorphic continuation elsewhere, where $0 < t \in \mathbf{R}$, $t^z := exp(zlnt)$, $z \in \mathcal{A}_v$, t^z take its principal value, dt corresponds to the Lebesgue measure on $\mathbf{R}$, $ln : (0, \infty) \to \mathbf{R}$ is the classical (natural) logarithmic function.

Denote by $Re(z) := (z + z^*)/2$ the real part of $z \in \mathcal{A}_v$, $\mathcal{I}_v := \{z \in \mathcal{A}_v : Re(z) = 0\}$, where z^* is the conjugate of a Cayley-Dickson number z.

2. Proposition. *The gamma function has as singularities only simple poles at the points $z \in \{0, -1, -2, ...\}$ and $res(-n, \Gamma)M = [(-1)^n/n!]M$ for each $M \in \mathcal{I}_v$.*

Proof. Write $\Gamma(z)$ in the form:

(i) $\Gamma(z) = \Phi(z) + \Psi(z)$, where

(ii) $\Phi(z) = \int_0^1 e^{-t} t^{z-1} dt$,

(iii) $\Psi(z) := \int_1^\infty e^{-t} t^{z-1} dt$.

Since $|e^{a+M}| = e^a$ for each $a \in \mathbf{R}$ and $M \in \mathcal{I}_v$ (see Corollary 3.3 [13]), then $|t^{z-1}| \le t^{\delta-1}$ for each $Re(z) \le \delta$, where $\delta > 0$ is a marked number. From

$\lim_{t\to\infty} e^{-t}t^{z-1} = 0$ it follows, that there exists $C = const > 0$ such that $|e^{-t}t^{z-1}| \le Ce^{-t/2}$ for each $t > 0$ and each z with $Re(z) \le \delta$. Therefore, $\Psi(z)$ is the $\mathcal{A}_v$-holomorphic function in $\mathcal{A}_v$.

Consider change of variables $t = 1/u$, then $\Phi(z) = \int_1^\infty e^{-1/u}u^{-z-1}du$ for each $\delta > 0$ and each z with $Re(z) \le \delta$, hence $|e^{-1/u}u^{-z-1}| \le u^{-\delta-1}$. Therefore, $\Phi(z)$ is $\mathcal{A}_v$-holomorphic, when $Re(z) > 0$. Substituting the Taylor series for e^{-t} into the integral expression (ii), we get

(iv) $\Phi(z) = \sum_{n=0}^\infty (-1)^n \int_0^1 t^{n+z-1}dt/n! = \sum_{n=0}^\infty (-1)^n (n+z)^{-1}/n!$. Series (iv) is uniformly and aboslutely convergent in any closed domain in $\mathcal{A}_v \setminus \{0,-1,-2,...\}$ and this series gives $\mathcal{A}_v$-analytic continuation of $\Phi(z)$. Thus $\Gamma(z)$ has only simple poles at the points $z \in \{0,-1,-2,...\}$.

The following Tannery lemma is true for $\mathcal{A}_v$-valued functions (for complex valued functions see §9.2 [3]).

3. Lemma. *If $g(t)$ and $f(t,n)$ are functions from $[a,\infty)$ to $\mathcal{A}_v$, $v \ge 2$, $\lim_{n\to\infty} f(t,n) = g(t)$, $\lim_{n\to\infty}\lambda_n = \infty$, then $\lim_{n\to\infty}\int_a^{\lambda_n} f(t,n)dt = \int_a^\infty g(t)dt$, provided that $f(t,n)$ tends to $g(t)$ uniformly on any fixed interval, and provided also that there exists a positive function $M(t)$ such that $|f(t,n)| \le M(T)$ for each values of n and t and such that $\int_a^\infty M(t)dt$ converges.*

Proof. The sequence $f(t,n)$ converges uniformly to $g(t)$ in the fixed segment $a \le t \le b$, $a < b < \infty$. Using triangle inequalities and $|\int_a^b g(t)dt| \le \int_a^b |g(t)|dt$ gives: $\limsup_{n\to\infty} |\int_a^{\lambda_n} f(t,n)dt - \int_a^\infty g(t)dt| \le 2\int_b^\infty M(t)dt$ for each $a < b < \infty$. From $\lim_{b\to\infty}\int_b^\infty M(t)dt = 0$ the statement of this lemma follows.

4. Proposition. *If $\Gamma(z,n) := n!n^z[(...((z(z+1))(z+2))...(z+n)]^{-1}$, $n \in \mathbf{N}$, then $\Gamma(z,n)$ tends to $\Gamma(z)$ as $n \to \infty$, the convergence being uniform in any bounded canonical closed subset $U \subset \mathcal{A}_v$ which contains no any of the singularities of $\Gamma(z)$, $v \ge 2$.*

Proof. Since $\mathcal{A}_v$ is power-associative and $\mathbf{R}$ is the centre of the Cayley-Dickson algebra, then $\{n^z[z(z+1)(z+2)...(z+n)]^{-1}\}_{q(n+2)}$ does not depend on the order of multiplication regulated by the vector $q(n+2)$ (see [13]). Therefore, $\Gamma(z,n) = (n/(n+1))^z z^{-1}\prod_{m=1}^n\{(1+1/m)^z(1+z/m)^{-1}\}$. Then $(1+1/m)^z(1+z/m)^{-1} = 1 + z(z-1)/(2m^2) + O(1/m^3)$, when $m > 0$ is large, hence $z^{-1}\prod_{m=1}^n\{(1+1/m)^z(1+z/m)^{-1}\}$ converges uniformly and aboslutely in any bounded canonical closed domain U in $\mathcal{A}_v$ to an $\mathcal{A}_v$-holomorphic function in accordance with Theorem 3.21 [13]. In view of Formulas $(3.6, 7)$ in [13] $|(1-t/n)^n t^{z-1}| = (1-t/n)^n t^{a-1} \le e^{-t}t^{a-1}$, where $a := Re(z)$. From

$\int_0^n (1-t/n)^n t^{z-1} dt = n^z \int_0^1 (1-u)^n u^{z-1} du$ and integrating by parts we get $\Gamma(z,n) = \int_0^n (1-t/n)^n t^{z-1} dt$, hence $\lim_{n\to\infty} \Gamma(z,n) = \int_0^\infty e^{-t} t^{z-1} dt =: \Gamma(z)$ by Lemma 3.

5. Remark. From the proof of Proposition 4 it follows, that $\Gamma(z) = z^{-1} \prod_{m=1}^\infty \{(1+1/m)^z (1+z/m)^{-1}\}$ for each $z \in \mathcal{A}_v \setminus \{0,-1,-2,...\}$, $v \ge 1$. The latter is known as the Euler's formula in the case of complex numbers.

6. Proposition. *The gamma function satisfies identities:*

(i) $\Gamma(z+1) = z\Gamma(z)$ *and*

(ii) $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$

for each $z \in \mathcal{A}_v \setminus \{0,-1,-2,...\}$, $v \ge 2$.

Proof. In view of power associativity of $\mathcal{A}_v$ and that $\mathbf{R}$ is the centre of the Cayley-Dickson algebra we get

$\Gamma(z+1) = z \lim_{n\to\infty} n! n^z [z(z+1)...(z+n)]^{-1} n/(z+n+1) = z\Gamma(z)$, also

$\Gamma(z)\Gamma(1-z) = \lim_{n\to\infty} \{z(1-z^2/1^2)(1-z^2/2^2)...(1-z^2/n^2)(1+(1-z)/n)\}^{-1} = \{z \prod_{n=1}^\infty (1-z^2/n^2)\}^{-1}$. In view of §6.83 $\pi \csc(\pi z) = \{z \prod_{n=1}^\infty (1-z^2/n^2)\}^{-1}$ for complex z in $\mathbf{C} \setminus \mathbf{Z}$ [3]. Using the $\mathcal{A}_v$-holomorphic extension of this function from the complex domain onto the corresponding domain $\mathcal{A}_v \setminus \mathbf{Z}$ (see Proposition 3.13, Corollary 2.13 and Theorems 3.10, 3.21 [13]), we get Formula *(ii)*.

7. Definition. A function F on an unbounded domain U in $\mathcal{A}_v$, $v \ge 2$, is said to have an asymptotic expansion $F \sim \sum_{|k|\le 0} (a_k, z^k)$, if

$\lim_{z\in U, |z|\to\infty} z^n \{F(z) - \sum_{|k|\le 0} (a_k, z^k)\} = 0$

for each $n \in \mathbf{N}$, where $k = (k_1, ..., k_n)$, $|k| := k_1 + ... + k_n$, $k_j \in \mathbf{Z}$ for each j, $n \in \mathbf{N}$, $(a_k, z^k) := a_{k_1} z^{k_1} ... a_{k_n} z^{k_n}$, $a_{k_j} \in \mathcal{A}_v$ for each j.

We write $F(z) \sim G(z) \sum_{|k|\le 0} (a_k, z^k)$, if $G(z)^{-1} F(z) \sim \sum_{|k|\le 0} (a_k, z^k)$. The term $G(z) a_0$ is called the dominant term of the asymptotic representation of $F(z)$.

8. Lemma. *Let $f(t)$ be a function in an unbounded domain U in $\mathcal{A}_v$ possibly with a branch point at* 0 *and such that*

$f(z) = \sum_{m=1}^\infty a_m z^{(m/r)-1}$,

when $|z| \le a$, $a > 0$, $r > 0$, let also f be $\mathcal{A}_v$-holomorphic in $B(U, 0, a+\delta) \setminus \{0\}$, where $\delta > 0$. Suppose, that when $t \ge 0$, $|f(t)| < Ce^{bt}$, where $C > 0$ and $b > 0$ are constants. Then

$F(z) = \int_0^\infty e^{-zt} f(t) dt \sim \sum_{n=1}^\infty a_n \Gamma(n/r) z^{-n/r}$,

when $|z|$ is large and $|Arg(z)| \le \pi/2 - \epsilon$, where $\epsilon > 0$ is arbitrary.

Proof. For each $n \in \mathbf{N}$ there exists a constant $C = const > 0$ such that

$|f(t)-\sum_{m=1}^{n-1}a_m t^{(m/r)-1}| \le Ct^{(n/r)-1}e^{bt}$

for each $t \ge 0$. In view of Formulas $(3.2, 3)$ [13]

$|\int_0^\infty e^{-zt}[f(t)-\sum_{m=1}^{n-1}a_m t^{(m/r)-1}]dt| \le \int_0^\infty e^{-xt}Ct^{(n/r)-1}e^{bt}dt$
$= C\Gamma(n/r)(x-b)^{-n/r}$

for each $x > b$, where $x := Re(z)$. From the condition $|Arg(z)| \le \pi/2-\epsilon$ it follows, that $x \ge |z|\sin(\epsilon)$, such that $x > b$ for $|z| > b\csc(\epsilon)$. Therefore, for $|Arg(z)| \le \pi/2-\epsilon < \pi/2$ and $|z| > b\csc(\epsilon)$, there is the inequality:
$|z^{n/r}\int_0^\infty e^{-zt}[f(t)-\sum_{m=1}^{n-1}a_m t^{(m/r)-1}]dt| \le C\Gamma(n/r)|z|^{n/r}/(|z|\sin(\epsilon)-b)^{n/r} = O(1)$.

9. Proposition. *Let $0 < \delta < \pi/2$, $z \in \mathcal{A}_v \setminus \{0,-1,-2,...\}$, $|Arg(z)| \le \pi-\delta$, $v \ge 2$. Then there exists the asymptotic expansion:*

$Ln\Gamma(z) \sim (z-1/2)Ln(z) - z + (ln(2\pi))/2 + \sum_{n=1}^\infty (-1)^{n-1}B_n[2n(2n-1)z^{2n-1}]^{-1}$,

where B_n are Bernoulli numbers defined by the equation: $(z/2)\coth(z/2) = 1+\sum_{n=1}^\infty(-1)^{n-1}B_n z^{2n}/(2n)!$.

Proof. If $z > 0$, then the substitution $t = zu$ gives $\Gamma(z) = \Gamma(1+z)/z = z^z e^{-z}\int_0^\infty (ue^{1-u})^z du$ and by analytic continuation the formula $\Gamma(z) = z^z e^{-z}\int_0^\infty(ue^{1-u})^z du$ is true for each copy of $\mathbf{C}$, $0 \in \mathbf{C}$, embedded into $\mathcal{A}_v$. In view of independence of this formula from such embedding and power associativity of $\mathcal{A}_v$ it follows, that it is true for each $z \in \mathcal{A}_v$ with $Re(z) > 0$. For $Re(z) > 0$ and large $|z|$ using substitutions $e^{-t} = \eta e^{1-\eta}$ for $t \in (0,\infty)$ and $\eta \in (1,\infty)$; also the substitution $e^{-t} = ue^{1-u}$ for t decreasing monotonously from ∞ to 0 and $u \in (0,1)$, we get $z^{-z}e^z\Gamma(z) = \int_0^\infty e^{-zt}(d\eta/dt - du/dt)dt$. Consider two real solutions η and u of the equation $t = u-1-ln(u)$ and the equation $\zeta^2/2 = w - Ln(1+w)$ which defines $w = w(\zeta)$ for $\zeta \in \mathbf{R}\oplus M\mathbf{R}$. It has two branches $\zeta = \beta w(1-2w/3+2w^2/4-...)^{1/2}$, where $\beta = -1$ or $\beta = 1$. Each branch is the analytic function of w in the domain $\{w \in \mathbf{R}\oplus M\mathbf{R} : |w| < 1\}$ with a simple zero at $w = 0$. For $\beta = 1$ there exists a unique solution $w = \zeta + a_2\zeta^2 + a_3\zeta^3 + ...$ in $\{\zeta : |\zeta| < \rho\}$, $na_nM = res(0,\zeta^{-n})M$ for each $n > 1$.

Thus w has two branches w_1 and $w_2(\zeta) = w_1(-\zeta)$. Singularities of $w(\zeta)$ are only points at which $dw/d\zeta$ is zero or infinite, hence these are $\zeta = 0$, also points corresponding to $w = 0$ and $w = -1$, since $dw/d\zeta = \zeta(1+w)/w$. Then $\zeta = 0$ is not a branch-point of w_1, to $w = -1$ there corresponds $\zeta = \infty$. Therefore, singularities are: $\zeta^2 = 4n\pi M$, where $n \in \mathbf{Z}\setminus\{0\}$. Then η and u are $\mathcal{A}_v$-holomorphic, when $|(z+\tilde{z})/2| < 2\pi$ possibly besides $z = 0$ and when $|z| < 2\pi$, where $\zeta^2 =: 2z$ such that $\eta = 1+(2z)^{1/2}+a_2(2z)+a_3(2z)^{3/2}+a_4(2z)^2+...$,

$u = 1-(2z)^{1/2}+a_2(2z)-a_3(2z)^{3/2}+a_4(2z)^2-...$, the square roots are taken positive, when $z>0$. Applying Lemma 8 we get the asympotic expansion. In view of Theorem 2.15 [13] for $M \in \mathcal{I}_v$ with $|M|=1$ and $\alpha \in \mathbf{R}$ and a loop defined by ρe^{Mt} on the boundary of the sector $|z| \le \rho$ and two lines $Arg(z)=0$, $Arg(z)=M\alpha$, where $\alpha \in (-\pi/2, \pi/2)$, $g(z) := d(\eta - u)/dt$, provides the equality: $\int_0^\infty e^{-zt}g(t)dt = \int_0^\infty \exp(-zte^{M\alpha})g(te^{M\alpha})e^{M\alpha}dt$, when $Arg(z) \in M\mathbf{R}$, $Re(z)>0$ and $Re(ze^{M\alpha})>0$, since $\mathbf{R} \oplus M\mathbf{R}$ is isomorphic with $\mathbf{C}$ which is commutative. Therefore, the latter integral converges uniformly and provides the analytic function. Two regions $Re(z)>0$ and $Re(ze^{M\alpha})>0$ have a common area and by the analytic continuation: $z^{-z}e^z\Gamma(z) = \int_0^\infty \exp(-zte^{M\alpha})g(te^{M\alpha})e^{M\alpha}dt$, when $\alpha \in (-\pi/2, \pi/2)$. Applying Lemma 8 we get the region of validity of this asymptotic expansion, since M is arbitrary.

10. Corollary. *For large $|y|$ there is the asymptotic expansion $|\Gamma(x+My)| \sim (2\pi)^{1/2}|y|^{x-1/2}\exp(-\pi|y|/2)$ uniformly by $M \in \mathcal{I}_v$, $|M|=1$, where $v \ge 2$, $y \in \mathbf{R}$.*

11. Corollary. $\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$ *for each $z \in \mathcal{A}_v \setminus \{0,-1,-2,...\}$, $v \ge 2$.*

The proof is analogous to §§9.55, 9.56 [3], since $\mathbf{R} \oplus M\mathbf{R}$ is isomorphic with $\mathbf{C}$ for each $M \in \mathcal{I}_v$, $v \ge 2$, $|M|=1$.

12. Proposition. *For all $z \in \mathcal{A}_v$:*

$1/\Gamma(z) = (2\pi)^{-1}(\int_\psi e^\zeta \zeta^{-z} d\zeta)M^*$

for a loop ψ and z in the plane $\mathbf{R} \oplus M\mathbf{R}$, $M \in \mathcal{I}_v$, $|M|=1$, ψ starts at $-\infty$ of the real axis, encircles 0 once in the positive direction and returns to the starting point.

Proof. Consider the integral $\int_\psi e^\zeta \zeta^{-z} d\zeta =: \int_\psi f(\zeta)d\zeta$, the integrand $f(\zeta)$ has a branch point at zero, but each branch is a one-valued function of ζ and each branch is $\mathcal{A}_v$-holomorphic in $\mathcal{A}_v \setminus Q$, where Q is a submanifold in $\mathcal{A}_v$ of real codimension 1 such that $(-\infty, 0] \subset Q$ (see §3.7 [12]). Then take a branch $e^\zeta \zeta^{-z} = \exp(\zeta - zLn(\zeta))$, where $Ln(\zeta)$ takes its principal value. Consider a rectifiable loop γ in $\mathbf{R} \oplus M\mathbf{R}$ encompassing zero in the positive direction and beginning at $-\rho$ on the lower edge of the cut and returns to $-\rho$ at the upper edge of the cut, where $\rho>0$.

In view of Theorem 2.15 [13] the value of the integral is not changed by the deformation to a contour γ consisting of the lower edge of the cut intersected with $[-\rho, -\delta]$, where $0<\delta<\rho$, the circle $|z|=\delta$ in the plane $\mathbf{R} \oplus M\mathbf{R}$, and the upper edge of the cut intersected with $[-\rho, -\delta]$. On

the upper edge of the cut in γ: $\zeta = ue^{\pi M}$, where $u > 0$, $u \in \mathbf{R}$, and $f(\zeta) = \exp(-u - zln(u) - z\pi M) = e^{-u}u^{-z}e^{-z\pi M}$. On the lower egdge of the cut in γ: $\zeta = ue^{-\pi M}$ and $f(\zeta) = e^{-u}u^{-z}e^{z\pi M}$. Therefore, $\int_\gamma f(\zeta)d\zeta = (e^{z\pi M} - e^{-z\pi M}) \int_\delta^\rho e^{-u}u^{-z}du + J$, where $J := \int_{-\pi}^{\pi} \exp(\delta e^{\theta M}) M \delta^{(1-z)} e^{(1-z)\theta M} d\theta$, since $\mathbf{R} \oplus M\mathbf{R}$ is isomorphic with $\mathbf{C}$ and $\hat{f}(z)|_{\mathbf{R}\oplus M\mathbf{R}}.h = f(z)h$ for each h and $z \in \mathbf{R} \oplus M\mathbf{R}$ (see Theorem 2.7 [13]).

If $z = x + yM$, where x and $y \in \mathbf{R}$, then $|J| \le \int_{-\pi}^{\pi} \delta^{1-x} \exp(\delta\cos(\theta) + y\theta)d\theta \le 2\pi\delta^{1-x}e^{\delta+\pi|y|}$, consequently, $\lim_{\delta\to 0} J = 0$, when $x < 1$. Hence $\int_\gamma e^{-\zeta}\zeta^{-z}d\zeta = 2\sin(\pi z)(\int_0^\rho e^{-u}u^{-z}dz)M$ for $Re(1-z) > 0$. Suppose ψ is the loop obtained from γ by tending ρ to the infinity, then $\int_\psi e^{\zeta}\zeta^{-z}d\zeta = 2\sin(\pi z)(\int_0^\infty e^{-u}u^{-z}du)M = 2\sin(\pi z)\Gamma(1-z)M$, since $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$, hence $1/\Gamma(z) = (2\pi)^{-1}(\int_\psi e^\zeta \zeta^{-z}d\zeta)M^*$. Since $M \in \mathcal{I}_v$ with $|M| = 1$ is arbitrary, then this formula is true in $\mathcal{A}_v \setminus Q$ with $Re(1-z) > 0$. By the complex holomorphic continuation this formula is true for all values of z in $\mathbf{R} \oplus M\mathbf{R}$.

13. Corollary. *Let M and ψ be as in Proposition 12, then* $\Gamma(z) = (2\sin(\pi z))^{-1}(\int_\psi e^\zeta\zeta^{z-1}d\zeta)M^*$ *for each* $z \in \mathbf{R} \oplus M\mathbf{R} \setminus \mathbf{Z}$.

14. Definition. The Beta function $B(p,q)$ of Cayley-Dickson numbers $p, q \in \mathcal{A}_v$, $v \ge 2$, is defined by the equation:

$B(p,q) := \int_0^1 \zeta^{p-1}(1-\zeta)^{q-1}d\zeta$,

whenever this integral (Eulerian of the first kind) converges, where $\zeta^{p-1} := e^{(p-1)ln(\zeta)}$ and the logarithm has its principal value. This equation defines $B(p,q)$ for each $Re(p) > 0$ and $Re(q) > 0$. For others values of p and q it is defined by the complex holomorphic continuation by p and q separately and subsequently in each complex plane $\mathbf{R} \oplus M\mathbf{R}$ and $\mathbf{R} \oplus S\mathbf{R}$, $M, S \in \mathcal{I}_v$, $|M| = 1$ and $|S| = 1$.

15. Proposition. *Let $p, q \in \mathcal{A}_v$, $v \ge 2$, such that the minimal subalgebra $\Upsilon_{p,q}$ containing p and q has embedding into $\mathbf{K}$, then*

$B(p,q) - B(q,p) = [B(p,q) - B(p, q_0 - q') - B(p_0 - p', q) + B(p_0 - p', q_0 - q')](q')^* {q'}_2/2$,

where $p_0 := Re(p)$, $p' := p - Re(p)$, ${q'}_2 \perp p'$, ${q'}_1 \parallel p'$ relative to the scalar product $(z, \eta) := Re(z\eta^*)$, $q' = {q'}_1 + {q'}_2$.

Proof. Making the substitution $\eta \mapsto 1 - \eta$ of the variable, we get $\int_0^1 \eta^{q-1}(1-\eta)^{p-1}d\eta = \int_0^1 (1-\eta)^{q-1}\eta^{p-1}d\eta$, but in general p and q do not commute. In view of Formulas (3.2, 3.3) [13] the commutator of two terms in the integral is:

$[t^{p-1}, (1-t)^{q-1}] = 2t^{p_0-1}(1-t)^{q_0-1}[(\sin|p'lnt|)/|p'lnt|)(\sin|q'ln(1-t)|)/|q'ln(1-$

$t)|)](p'lnt)(q'_2ln(1-t))$.
On the other hand, $[(\sin|M|)/|M|]M = [e^M - e^{-M}]/2$ for each $M \in \mathcal{I}_v$, hence $\int_0^1 [t^{p-1}, (1-t)^{q-1}]dt = (\int_0^1 t^{p_0-1}(1-t)^{q_0-1}[t^{p'} - t^{-p'}][(1-t)^{q'} - (1-t)^{-q'}]dt)(q')^* {q'}_2/2$ $= [B(p,q) - B(p, q_0 - q') - B(p_0 - p', q) + B(p_0 - p', q_0 - q')](q')^* {q'}_2/2$, since $\mathbf{K}$ is alternative and $p'{q'}_2 = p'((q'{q'}^*){q'}_2) = p'(q'({q'}^*{q'}_2))$.

16. Remark. Let G be a classical Lie group over $\mathbf{R}$ and $g = T_eG$ be its Lie algebra (finite dimensional over $\mathbf{R}$). Suppose that $e : V \to U$ is the exponential mapping of the neighbourhood V of zero in g into a neighbourhood U of the unit element $e \in G$, $ln : U \to V$ is the logarithmic mapping. Then $w = ln(e^u \circ e^v)$, $w = w(u,v)$, is given by the Campbell-Hausdorff formula in terms of the adjoint representation $(ad \quad u)(v) := [u,v]$:

$$w = \sum_{n=1}^{\infty} n^{-1} \sum_{r+s=n, r\ge 0, s\ge 0} ({w'}_{r,s} + {w"}_{r,s}), \text{ where } {w'}_{r,s} = \sum_{m=1}^{\infty}$$

$$(-1)^{m-1} m^{-1} \sum^{*} ((\prod_{i=1}^{m-1} (ad \quad u)^{r_i} (ad \quad v)^{s_i} (r_i!)^{-1} (s_i!)^{-1})(ad \quad u)^{r_m} (r_m!)^{-1})(v),$$

$${w"}_{r,s} = \sum_{m=1}^{\infty} (-1)^{m-1} m^{-1} \sum^{**} ((\prod_{i=1}^{m-1} (ad \quad u)^{r_i} (ad \quad v)^{s_i} (r_i!)^{-1} (s_i!)^{-1})(u),$$

$\sum^*$ means the sum by $r_1 + ... + r_m = r$, $s_1 + ... + s_{m-1} = s - 1$, $r_1 + s_1 \ge 1$, ..., $r_{m-1} + s_{m-1} \ge 1$, $\sum^{**}$ means the sum by $r_1 + ... + r_{m-1} = r - 1$, $s_1 + ... + s_{m-1} = s$, $r_1 + s_1 \ge 1$, ..., $r_{m-1} + s_{m-1} \ge 1$. In particular, this formula can be applied to the multiplicative group $G = \mathbf{H} \setminus \{0\}$ with $U = G$ and $V = g$, since each quaternion can be represented as a 2×2 complex matrix, where generators of $\mathbf{H}$ are Pauli matrices [2].

17. Theorem. *Let $p, q \in \mathcal{A}_v$, $v \ge 2$, such that the minimal subalgebra $\Upsilon_{p,q}$ generated by p and q has embedding into $\mathbf{H}$, then*
$\Gamma(p)\Gamma(q) = \Gamma(w(p,q))B(p,q) -$
$[\Gamma(w(p,q)) - \Gamma(w(p, q_0 - q'))]{q'}^*{q'}_2[B(p,q) - B(p_0 - p', q)]/2$,
where $p_0 := Re(p)$, $p' := p - Re(p)$, ${q'}_2 \perp p'$, ${q'}_1 \parallel p'$ relative to the scalar product $(z,\eta) := Re(z\eta^*)$, $q' = {q'}_1 + {q'}_2$, $w(p,q)$ is given in Remark 16.

Proof. Let $S_R := \{(x,y) \in \mathbf{R^2} : 0 \le x \le R, 0 \le y \le R\}$. Then
$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-x} x^{p-1} dx \int_0^\infty e^{-y} y^{q-1} dy$
$= \lim_{R\to\infty} \int_0^R (\int_0^R e^{-x-y} x^{p-1} y^{q-1} dy) dx$
$= \lim_{R\to\infty} \int\int_{S_R} e^{-x-y} x^{p-1} y^{q-1} dx dy$
in accordance with the Fubini theorem, since each function $f : U \to \mathbf{H}$ has

the form $f(z) = f_1(z) + f_i(z)i + f_j(z)j + f_k(z)k$ for each z in a domain U in $\mathbf{H^n}$, f_1, f_i, f_j, f_k are real-valued functions, $\{1, i, j, k\}$ are generators of $\mathbf{H}$. Consider a triangle $T_R := \{(x, y) \in \mathbf{R^2} : 0 \le x, 0 \le y, x + y \le R\}$ and put $f(x, y) := e^{-x-y}x^{p-1}y^{q-1}$, where $p, q \in \mathcal{A}_v$ are marked, then
$|\int\int_{S_R} f(x, y)dxdy - \int\int_{T_R} f(x, y)dxdy| \le \int\int_{S_R\setminus T_R} |f(x, y)|dxdy$
$\le \int\int_{S_R} |f(x, y)|dxdy - \int\int_{S_{R/2}} |f(x, y)|dxdy.$
We have $\lim_{R\to\infty} \int\int_{S_R} |f(x, y)|dxdy = \Gamma(p_0)\Gamma(q_0)$, hence
$\lim_{R\to\infty} \int\int_{S_R\setminus S_{R/2}} |f(x, y)|dxdy = 0$. Therefore,
$\Gamma(p)\Gamma(q) = \lim_{R\to\infty} \int\int_{T_R} e^{-x-y}x^{p-1}y^{q-1}dxdy.$
The substitution $x + y = \xi$, $y = \xi\eta$ and application of the Fubini theorem gives
$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^1 e^{-\xi}\xi^p(1 - \eta)^{p-1}\xi^{q-1}\eta^{q-1}d\xi d\eta,$
since $\mathbf{H}$ is associative, ξ^{p-1} commutes with $(1 - \eta)^{p-1}$, ξ^{q-1} commutes with η^{q-1}. Therefore,
$\Gamma(p)\Gamma(q) = \Gamma(w(p, q))B(p, q) + \int_0^\infty \int_0^1 e^{-\xi}\xi^p[(1 - \eta)^{p-1}, \xi^{q-1}]\eta^{q-1}d\xi d\eta.$
Let M and N be in $\mathcal{I}_v$, then $e^N e^M = (\cos|M|)e^N + [(\sin|N|)/|N|]Me^{N_1-N_2}$, where $M \perp N_2$, $M \parallel N_1$ relative to the scalar product $(z, \eta) := Re(z\eta^*)$, $N_1, N_2 \in \mathcal{I}_v$, $N = N_1 + N_2$ (see Formulas (3.2, 3.3) [13]). Therefore,
$\int_0^\infty \int_0^1 e^{-\xi}\xi^p[(1 - \eta)^{p-1}, \xi^{q-1}]\eta^{q-1}d\xi d\eta =$
$-\int_0^\infty \int_0^1 e^{-\xi}\xi^{p-1}[\xi^q - \xi^{q_0-q'}](q'^*q'_2)[(1 - \eta)^{p-1} - (1 - \eta)^{p_0-p'-1}]\eta^{q-1}d\xi d\eta/2$
$= -[\Gamma(w(p, q)) - \Gamma(w(p, q_0 - q'))](q'^*q'_2)[B(p, q) - B(p_0 - p', q)]/2.$

18. Note. Proposition 15 and Theorem 17 show differences in identities for Beta and Gamma functions between commutative case of $\mathbf{C}$ and noncommutative cases of $\mathcal{A}_v$, $v \ge 2$, and $\mathbf{H}$ particularly. Certainly, in the particular case if $\Upsilon_{p,q}$ has embedding into $\mathbf{C}$, then $q'^*q'_2 = 0$ and Proposition 15 and Theorem 17 give classical results, but for general p and q the subalgebra $\Upsilon_{p,q}$ can have no any embedding into $\mathbf{C}$.

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Address: Sergey V. Ludkovsky, Mathematical Department, TW-WISK, Brussels University, V.U.B., Pleinlaan 2, Brussels 1050, Belgium.
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